PSEUDO-RIEMANNIAN MANIFOLDS WITH COMMUTING JACOBI OPERATORS

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Abstract. We study the geometry of pseudo-Riemannian manifolds which are Jacobi–Tsankov, i.e. $\mathcal{J}(x)\mathcal{J}(y)=\mathcal{J}(y)\mathcal{J}(x)$ for all x,y. We also study manifolds which are 2-step Jacobi nilpotent, i.e. $\mathcal{J}(x)\mathcal{J}(y)=0$ for all x,y.

1. Introduction

Let $\mathcal{M}:=(M,g)$ be a pseudo-Riemannian manifold of signature (p,q) and dimension $m=p+q\geq 3$; $\mathcal{M}$ is said to be *Riemannian* if $p=0$ and *Lorentzian* if $p=1$. Although the Riemannian and Lorentzian settings are perhaps the most frequently studied, pseudo-Riemannian manifolds with other signatures are important in many physical applications; see, for example, the discussion of Kaluza-Klein gravity in Overduin and Wesson [15] or the brane world cosmology of Shtanov and Sahni [16]. Thus the higher signature setting is important not only mathematically, but also in physical applications.

Let $\mathcal{R}$ be the curvature operator and $\mathcal{J}$ the Jacobi operator which are defined by the Levi-Civita connection on $\mathcal{M}$:

$$\begin{aligned}&\mathcal{R}(x,y):=\nabla_x\nabla_y-\nabla_y\nabla_x-\nabla_{[x,y]},\\&\mathcal{J}(x):y\to\mathcal{R}(y,x)x\,.\end{aligned}$$

The relationship between the spectral geometry of $\mathcal{J}$ and the underlying geometry of the manifold has been studied extensively in recent years. Suppose that $\mathcal{M}$ is Riemannian. If $\mathcal{M}$ is a 2-point homogeneous space, then the group of isometries acts transitively on the unit sphere bundle $S(\mathcal{M})$ and hence the eigenvalues of $\mathcal{J}$ are constant on $S(\mathcal{M})$. Osserman [14] wondered if the converse is true, at least locally. He conjectured that if $\mathcal{M}$ is a Riemannian manifold such that the eigenvalues of $\mathcal{J}$ are constant on $S(\mathcal{M})$, then either $\mathcal{M}$ is flat or $\mathcal{M}$ is locally isometric to a rank 1-symmetric space. This conjecture has been established in dimensions $m\neq 16$ by the work of Chi [6] and Nikolayevsky [12, 13]; the case $m=16$ is still open.

Let $S^{\pm}(\mathcal{M})$ be the pseudo-sphere bundles of unit spacelike $(+)$ or unit timelike $(-)$ vectors. One says that a pseudo-Riemannian manifold $\mathcal{M}$ is *spacelike Osserman* (resp. *timelike Osserman*) if the eigenvalues of the Jacobi operator $\mathcal{J}$ are constant on $S^+(\mathcal{M})$ (resp. on $S^-(\mathcal{M})$). Work of García–Río et. al. [8] shows these are equivalent concepts so one simply speaks of an Osserman manifold. It is known [2, 8] that any Lorentzian Osserman manifold has constant sectional curvature; thus the geometry is very rigid in this setting. However if $p\geq 2$ and $q\geq 2$, there are Osserman pseudo-Riemannian manifolds which are not locally homogeneous; see, for example, [3, 7].

One can weaken this condition slightly. Let $p\geq 1$ and $q\geq 1$. One says that $\mathcal{M}$ is *pointwise Osserman* if the spectrum of $\mathcal{J}$ is constant on $S_P^+(\mathcal{M})$, or equivalently on $S_P^-(\mathcal{M})$, for every $P\in M$. Blažić [1] has shown that if the spectrum of $\mathcal{J}$ is bounded on either $S_P^+(\mathcal{M})$ or, equivalently, $S_P^-(\mathcal{M})$, for every $P\in M$, then necessarily $\mathcal{M}$ is *pointwise Osserman*.

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In this paper, instead of focusing on the spectrum, we will relate commutativity properties of $\mathcal{J}$ to the underlying geometry.

Definition 1.1. One says that a pseudo-Riemannian manifold $\mathcal{M}$ is:

(1) 2-*step Jacobi nilpotent* if $\mathcal{J}(x)\mathcal{J}(y) = 0$ for all tangent vectors x, y.
(2) *Jacobi–Tsankov* if $\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$ for all tangent vectors x, y.
(3) *Orthogonally Jacobi–Tsankov* if $\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$ for all $x \perp y$.

Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3). The following seminal result was established by Tsankov [17]:

Theorem 1.2. *Let $\{\lambda_i\}$ be the eigenvalues of the shape operator of a hypersurface M in R^{m+1}. Then M is orthogonally Jacobi–Tsankov if and only if either $\lambda_1 = ... = \lambda_m$ or $\lambda_1 = ... = \lambda_{m-1} = 0,\ \lambda_m \neq 0$.*

Theorem 1.2 has been extended from hypersurfaces to the more general setting in [5]:

Theorem 1.3. *Let $\mathcal{M}$ be an orthogonally Jacobi–Tsankov Riemannian manifold. Then $\mathcal{M}$ has constant sectional curvature.*

In passing to more general signatures, we shall impose a stronger condition and study Jacobi–Tsankov manifolds. It is convenient to work in the algebraic context. Let V be a finite dimensional real vector space. Let $\mathfrak{A}(V) \subset \otimes^4 V^*$ be the space of *algebraic curvature tensors*; these are the 4-tensors with the same symmetries as the Riemann curvature tensor. Thus $A \in \mathfrak{A}(V)$ if and only if we have the following symmetries for all $x, y, z, w \in V$:

$$\begin{aligned}
&A(x,y,z,w) = -A(y,x,z,w) = A(z,w,x,y),\\
&A(x,y,z,w) + A(y,z,x,w) + A(z,x,y,w) = 0\,.
\end{aligned}$$

Let $\mathfrak{M} := (V, \langle\cdot,\cdot\rangle, A)$ where $A \in \mathfrak{A}(V)$ and where $\langle\cdot,\cdot\rangle$ is a non-degenerate symmetric bilinear form of signature (p,q) on V which is used to raise and lower indices. The corresponding *algebraic curvature operator* $\mathcal{A} \in V^* \otimes V^* \otimes \operatorname{End}(V)$ is characterized by

$$\langle \mathcal{A}(x,y)z, w\rangle = A(x,y,z,w)$$

and the Jacobi operator $\mathcal{J} = \mathcal{J}_A$ is given by $\mathcal{J}(x) : y \to \mathcal{A}(y,x)x$. The notions of Definition 1.1 then extend to the algebraic setting. In Section 2, we will show that:

Theorem 1.4. *Let $\mathfrak{M}$ be Jacobi–Tsankov. Then:*

(1) *$\mathcal{J}(x)^2 = 0$ for all $x \in V$.*
(2) *$\mathfrak{M}$ is Osserman.*
(3) *If V is Riemannian or Lorentzian, then $A = 0$.*

We can draw the following geometrical consequence from Theorem 1.4:

Corollary 1.5. *Let $\mathcal{M}$ be a Jacobi–Tsankov pseudo-Riemannian manifold of signature (p,q). Then $\mathcal{M}$ is nilpotent Osserman. If $p = 0$ or if $p = 1$, then $\mathcal{M}$ is flat.*

One might conjecture that the condition $\mathcal{J}(x)^2 = 0$ for all $x \in V$ is sufficient to imply $\mathfrak{M}$ is Jacobi–Tsankov. This is in fact not the case as we will show in Lemma 2.2.

It is clear that any 2-step Jacobi nilpotent algebraic curvature tensor is Jacobi–Tsankov. In Section 3, we will show that the converse holds in low dimensions:

Theorem 1.6. *Let $\mathfrak{M}$ be Jacobi–Tsankov. If $\dim(V) \leq 13$, then $\mathfrak{M}$ is 2-step Jacobi nilpotent.*

The condition $\dim(V)\le 13$ in Theorem 1.6 is sharp. In Lemma 3.2, we construct a Jacobi–Tsankov tensor in signature $(8,6)$, which is indecomposable and for which there exist (x,y) so that $\mathcal{J}(x)\mathcal{J}(y)\ne 0$.

There are similar questions for the skew-symmetric curvature operator.

Definition 1.7. One says that $\mathfrak{M}$ is:

(1) 2-*step skew-curvature nilpotent* if $\mathcal{A}(x_1,x_2)\mathcal{A}(x_3,x_4)=0$ for all tangent vectors x_1,x_2,x_3,x_4.
(2) *Skew–Tsankov* if $\mathcal{A}(x_1,x_2)\mathcal{A}(x_3,x_4)=\mathcal{A}(x_3,x_4)\mathcal{A}(x_1,x_2)$ for all tangent vectors x_1,x_2,x_3,x_4.

Motivated by Theorem 1.6, in Section 4, we will study 2-step Jacobi nilpotent algebraic curvature tensors in relation to 2-step skew-curvature nilpotent ones. If $A_W\in\mathfrak{A}(W)$, we say that (W,A_W) is *indecomposable* if there is no decomposition $(W,A_W)=(W_1,A_1)\oplus(W_2,A_2)$ where $\dim(W_i)\ge 1$. Similarly, we say that $\mathfrak{M}$ is indecomposable if there is no decomposition $\mathfrak{M}=\mathfrak{M}_1\oplus\mathfrak{M}_2$ so that $\dim(V_i)\ge 1$.

Definition 1.8. Let $A_W\in\mathfrak{A}(W)$. Assume that (W,A_W) is indecomposable. Let $\{\bar e_1,...,\bar e_k\}$ be a basis for an auxiliary vector space $\bar W$. Let

$$\begin{aligned}&\mathfrak{M}:=(W\oplus\bar W,\langle\cdot,\cdot\rangle_{W\oplus\bar W},A_W\oplus 0)\quad\text{where}\\&\langle e_i,e_j\rangle=\langle\bar e_i,\bar e_j\rangle=0,\quad\langle e_i,\bar e_j\rangle=\delta_{ij}\,.\end{aligned}\tag{1.a}$$

We will establish the following classification theorem:

Theorem 1.9. *The following statements are equivalent:*

(1) $\mathfrak{M}$ *is* 2-*step Jacobi nilpotent and indecomposable,*
(2) $\mathfrak{M}$ *is* 2-*step skew-curvature nilpotent and indecomposable,*
(3) $\mathfrak{M}$ *is isomorphic to one of the tensors described in Definition 1.8.*

One has the following geometrical examples which arose in the study of Osserman manifolds. We refer to [10, 11] for further details.

Theorem 1.10. *Let* $(x_1,...,x_p,y_1,...,y_p)$ *be coordinates on* $\mathbb{R}^{2p}$ *for* $p\ge 2$. *Let* $\psi_{ij}(x)=\psi_{ji}(x)$ *be a symmetric* 2-*tensor. Let*

$$g_\psi(\partial_{x_i},\partial_{x_j})=\psi_{ij}(x),\quad g_\psi(\partial_{x_i},\partial_{y_j})=\delta_{ij},\quad g_\psi(\partial_{y_i},\partial_{y_j})=0\,.$$

Then $\mathcal{M}:=(\mathbb{R}^{2p},g_\psi)$ *is a complete pseudo-Riemannian manifold of neutral signature* (p,p) *which is* 2-*step Jacobi nilpotent and* 2-*step skew-curvature nilpotent.*

2. The proof of Theorem 1.4

The Jacobi operator is quadratic in x. We polarize to define an operator valued bilinear form by setting:

$$\mathcal{J}(x,y):z\to\tfrac12\partial_\varepsilon\mathcal{J}(x+\varepsilon y)z\big|_{\varepsilon=0}=\tfrac12\{\mathcal{A}(z,x)y+\mathcal{A}(z,y)x\}\,.$$

Setting $x=y$ yields $\mathcal{J}(x,x)=\mathcal{J}(x)$. Furthermore

$$\mathcal{J}(x,y)x=\tfrac12(\mathcal{A}(x,x)y+\mathcal{A}(x,y)x)=-\tfrac12\mathcal{J}(y)x\,.$$

Let A be a Jacobi–Tsankov algebraic curvature tensor. Polarizing the identity $\mathcal{J}(x)\mathcal{J}(y)=\mathcal{J}(y)\mathcal{J}(x)$ yields:

$$\mathcal{J}(x_1,x_2)\mathcal{J}(y_1,y_2)=\mathcal{J}(y_1,y_2)\mathcal{J}(x_1,x_2)\,.$$

We have $\mathcal{J}(x)x=\mathcal{A}(x,x)x=0$. We prove Assertion (1) by computing:

$$0=\mathcal{J}(x,y)\mathcal{J}(x,x)x=\mathcal{J}(x,x)\mathcal{J}(x,y)x=-\tfrac12\mathcal{J}(x)\mathcal{J}(x)y\,.$$

Since the Jacobi operator is nilpotent, $\{0\}$ is the only eigenvalue of $\mathcal{J}$. This shows that A is Osserman.

If $p=0$, then $\mathcal{J}(x)$ is diagonalizable. Consequently, $\mathcal{J}(x)^2=0$ implies $\mathcal{J}(x)=0$ for all x. It now follows $A=0$. If $p=1$, then A is Osserman implies A has

constant sectional curvature [2, 8]. Since $\mathcal{J}(x)^2 = 0$, this again implies $A = 0$. This completes the proof of Theorem 1.4. $\square$

In fact, it is possible to work in a slightly more general setting. Following Bokan [4], one says that $\mathcal{C}$ is a *generalized curvature operator* if it has the symmetries of the curvature operator defined by a torsion free connection, i.e. if

$$\begin{aligned}&\mathcal{C}(x,y)z = -\mathcal{C}(y,x)z,\\&\mathcal{C}(x,y)z + \mathcal{C}(y,z)x + \mathcal{C}(z,x)y = 0.\end{aligned}$$

The proof given above then generalizes immediately to yield:

Corollary 2.1. *If $\mathcal{C}$ is a generalized curvature operator on V which is Jacobi–Tsankov, then $\mathcal{J}_C$ is Osserman and $\mathcal{J}_C(x)^2 = 0$ for all $x \in V$.*

Let ϕ be a skew-symmetric endomorphism of V. Define

$$A_\phi(x,y,z,w) := \langle \phi y, z\rangle\langle \phi x, w\rangle - \langle \phi x, z\rangle\langle \phi y, w\rangle - 2\langle \phi x, y\rangle\phi z, w\rangle\,.$$

The associated Jacobi operator is then given by

$$\mathcal{J}_\phi(x)y = -3\langle y, \phi x\rangle\phi x\,.$$

In the following example, we exhibit an algebraic curvature tensor so that $\mathcal{J}(x)^2 = 0$ for all $x \in V$, but which is not Jacobi–Tsankov. Let $\mathbb{R}^{(p,q)}$ denote Euclidean space with a metric of signature (p, q).

Lemma 2.2.

(1) *There exist skew-symmetric endomorphisms $\{\phi_1, \phi_2\}$ of $\mathbb{R}^{(4,4)}$ so that*

$$\phi_1^2 = \phi_2^2 = \phi_1\phi_2 + \phi_2\phi_2 = 0, \quad \textit{and} \quad \phi_1\phi_2 \neq 0\,.$$

(2) *Set $A = -\frac{1}{3}\{A_{\phi_1} + A_{\phi_2}\}$. Then $\mathcal{J}_A(x)^2 = 0$ for all x. Furthermore, A is not Jacobi–Tsankov.*

Proof. We apply Lemma 1.4.5 of [9] to find a collection $\{e_1, e_2, e_3, e_4\}$ of skew-symmetric endomorphisms of $\mathbb{R}^{(4,4)}$ so that:

$$e_1^2 = e_2^2 = \mathrm{id}, \quad e_3^2 = e_4^2 = -\,\mathrm{id}, \quad e_ie_j + e_je_i = 0 \text{ for } i \neq j\,.$$

Set $\phi_1 = e_1 + e_3$, $\phi_2 = e_2 + e_4$. These are skew-symmetric endomorphisms with

$$\phi_1^2 = \phi_2^2 = 0, \quad \phi_1\phi_2 + \phi_2\phi_1 = 0\,.$$

Suppose that

$$\alpha := \phi_1\phi_2 = (e_1 + e_3)(e_2 + e_4) = 0\,.$$

We argue for a contradiction. Conjugating by e_1 yields

$$e_1\alpha e_1 = (-e_1 + e_3)(e_2 + e_4) = 0\,.$$

Adding this equation to the previous one implies $e_3(e_2 + e_4) = 0$. Multiplying by e_3 implies $e_2 + e_4 = 0$. Conjugating this identity by e_2 yields $e_2 - e_4 = 0$ and thus $e_2 = 0$. This is not possible. Assertion (1) now follows.

To prove Assertion (2), we compute:

$$\begin{aligned}\mathcal{J}_A(x)y &= \langle y, \phi_1 x\rangle\phi_1 x + \langle y, \phi_2 x\rangle\phi_2 x,\\ \mathcal{J}_A(x_1)\mathcal{J}_A(x_2)y &= \langle y, \phi_1 x_2\rangle\langle \phi_1 x_2, \phi_1 x_1\rangle\phi_1 x_1 + \langle y, \phi_1 x_2\rangle\langle \phi_1 x_2, \phi_2 x_1\rangle\phi_2 x_1\\ &\quad + \langle y, \phi_2 x_2\rangle\langle \phi_2 x_2, \phi_1 x_1\rangle\phi_1 x_1 + \langle y, \phi_2 x_2\rangle\langle \phi_2 x_2, \phi_2 x_1\rangle\phi_2 x_1\\ &= \langle y, \phi_1 x_2\rangle\langle \phi_1 x_2, \phi_2 x_1\rangle\phi_2 x_1 + \langle y, \phi_2 x_2\rangle\langle \phi_2 x_2, \phi_1 x_1\rangle\phi_1 x_1\,.\end{aligned}$$

Since

$$\langle \phi_1 x, \phi_2 x\rangle = -\langle \phi_2\phi_1 x, x\rangle = \langle \phi_1\phi_2 x, x\rangle = -\langle \phi_2 x, \phi_1 x\rangle,$$

we have $\mathcal{J}(x)\mathcal{J}(x) = 0$ as desired.

Choose x_1 so $\phi_2\phi_1x_1\neq0$. Set $y=\phi_1x_1$. We then have:

$$\begin{array}{rcl}\mathcal{J}_A(x_1)\mathcal{J}_A(x_2)y&=&\langle\phi_1x_1,\phi_1x_2\rangle\langle\phi_1x_2,\phi_2x_1\rangle\phi_2x_1\\&+&\langle\phi_1x_1,\phi_2x_2\rangle\langle\phi_2x_2,\phi_1x_1\rangle\phi_1x_1\\&=&\langle\phi_1x_1,\phi_2x_2\rangle^2\phi_1x_1,\\\mathcal{J}_A(x_2)\mathcal{J}_A(x_1)y&=&\langle\phi_1x_1,\phi_1x_1\rangle\langle\phi_1x_1,\phi_2x_2\rangle\phi_2x_2\\&+&\langle\phi_1x_1,\phi_2x_1\rangle\langle\phi_2x_1,\phi_1x_2\rangle\phi_1x_2\\&=&0\,.\end{array}$$

Choose x_2 so $\langle\phi_1x_1,\phi_2x_2\rangle\neq0$. Then $\mathcal{J}_A(x_1)\mathcal{J}_A(x_2)y\neq0=\mathcal{J}_A(x_2)\mathcal{J}_A(x_1)y$. □

3. 2-step Jacobi nilpotent algebraic curvature tensors

Theorem 1.6 will follow from the following result:

Lemma 3.1. *Let $\mathfrak{M}:=(V,\langle\cdot,\cdot\rangle,A)$ be Jacobi–Tsankov. Suppose that there exist $x,y\in V$ so that $\mathcal{J}(x)\mathcal{J}(y)\neq0$.*

1. *There exists $w\in V$ so that*
$$\langle\mathcal{J}(x)\mathcal{J}(y)w,w\rangle=\langle\mathcal{J}(y)\mathcal{J}(w)x,x\rangle=\langle\mathcal{J}(w)\mathcal{J}(x)y,y\rangle\neq0\,.$$
2. *Let $\mathcal{J}_x:=\mathcal{J}(x)$, $\mathcal{J}_y:=\mathcal{J}(y)$ and $\mathcal{J}_{xy}:=\mathcal{J}(x,y)$. Set*
$$\begin{array}{llll}e_2=\mathcal{J}_x\mathcal{J}_yw,& e_3=\mathcal{J}_xw,& e_4=\mathcal{J}_yw,& e_5=\mathcal{J}_{xy}w\\f_2=\mathcal{J}_y\mathcal{J}_wx,& f_3=\mathcal{J}_yx,& f_4=\mathcal{J}_wx,& f_5=\mathcal{J}_{yw}x\\g_2=\mathcal{J}_w\mathcal{J}_xy,& g_3=\mathcal{J}_wy,& g_4=\mathcal{J}_xy,& g_5=\mathcal{J}_{wx}y.\end{array}$$
The set $S:=\{w,x,y,e_2,...,e_5,f_2,...,f_5,g_2,...,g_4\}$ is linearly independent.
3. $e_5+f_5+g_5=0$.
4. $\dim(V)\geq14$.

Proof. Choose w so that $e_2:=\mathcal{J}(x)\mathcal{J}(y)w\neq0$. Choose f so $\langle e_2,f\rangle\neq0$. Set $w(\varepsilon):=w+\varepsilon f$ and $e_2(\varepsilon):=\mathcal{J}(x)\mathcal{J}(y)w(\varepsilon)$. Then

$$p(\varepsilon):=\langle w(\varepsilon),e_2(\varepsilon)\rangle=\langle w,e_2\rangle+2\varepsilon\langle e_2,f\rangle+\varepsilon^2\langle\mathcal{J}(x)\mathcal{J}(y)f,f\rangle\,.$$

As $\langle e_2,f\rangle\neq0$, $p(\varepsilon)$ is a non-trivial polynomial in ε. Thus it is non-zero for a suitable choice of ε. Thus we may choose w so that $\langle w,\mathcal{J}(x)\mathcal{J}(y)w\rangle\neq0$. Now,

$$\begin{array}{rcl}\langle\mathcal{J}(y)\mathcal{J}(w)x,x\rangle&=&-2\langle\mathcal{J}(y)\mathcal{J}(w,x)w,x\rangle=-2\langle\mathcal{J}(y)w,\mathcal{J}(w,x)x\rangle\\&=&\langle\mathcal{J}(y)w,\mathcal{J}(x)w\rangle=\langle\mathcal{J}(x)\mathcal{J}(y)w,w\rangle\,.\end{array}$$

Similarly, $\langle\mathcal{J}(w)\mathcal{J}(x)y,y\rangle=\langle\mathcal{J}(x)\mathcal{J}(y)w,w\rangle$ and Assertion (1) follows.

Because $\mathcal{J}(x+\varepsilon y)\mathcal{J}(x+\varepsilon y)=0$ for every $\varepsilon\in\mathbb{R}$ and because $\mathfrak{M}$ is Jacobi–Tsankov, we have the following relations:

$$\begin{array}{lll}\mathcal{J}_x^2=0,&\mathcal{J}_y^2=0,&\mathcal{J}_x\mathcal{J}_y=\mathcal{J}_y\mathcal{J}_x,\\\mathcal{J}_x\mathcal{J}_{xy}=\mathcal{J}_{xy}\mathcal{J}_x=0,&\mathcal{J}_y\mathcal{J}_{xy}=\mathcal{J}_{xy}\mathcal{J}_y=0,&\mathcal{J}_{xy}^2=-\frac12\mathcal{J}_x\mathcal{J}_y\,.\end{array}$$

We have $\mathcal{J}_w\mathcal{J}_yx\neq0$ and $\mathcal{J}_w\mathcal{J}_xy\neq0$ by Assertion (1). To prove Assertion (2), suppose there is a non-trivial dependence relation among the elements of S:

$$\begin{array}{rcl}0&=&a_1w+a_2e_2+a_3e_3+a_4e_4+a_5e_5\\&+&b_1x+b_2f_2+b_3f_3+b_4f_4+b_5f_5\\&+&c_1y+c_2g_2+c_3g_3+c_4g_4+c_5g_5\\&=&a_1w+a_2\mathcal{J}_x\mathcal{J}_yw+a_3\mathcal{J}_xw+a_4\mathcal{J}_yw+a_5\mathcal{J}_{xy}w\\&+&b_1x+b_2\mathcal{J}_y\mathcal{J}_wx+b_3\mathcal{J}_yx+b_4\mathcal{J}_wx+b_5\mathcal{J}_{yw}x\\&+&c_1y+c_2\mathcal{J}_w\mathcal{J}_xy+c_3\mathcal{J}_wy+c_4\mathcal{J}_xy+c_5\mathcal{J}_{wx}y\,.\end{array}\tag{3.a}$$

Since we are **not** taking g_5, we must set

$$c_5=0\,.\tag{3.b}$$

We can apply $J_x J_y$ to Equation (3.a) to see $a_1 e_5 = 0$. Since $e_5 \neq 0$, $a_1 = 0$. Similarly $b_1 = c_1 = 0$. If we now apply J_x to Equation (3.a), we see

$$\begin{aligned} & a_4 \mathcal{J}_x \mathcal{J}_y w + c_3 \mathcal{J}_x \mathcal{J}_w y = 0 \quad \text{so} \\ & 0 = \langle a_4 \mathcal{J}_x \mathcal{J}_y w + c_3 \mathcal{J}_x \mathcal{J}_w y, w \rangle = a_4 \langle \mathcal{J}_x \mathcal{J}_y w, w \rangle \,. \end{aligned}$$

Since $\langle \mathcal{J}_x \mathcal{J}_y w, w \rangle \neq 0$, $a_4 = 0$. Similarly, we get $a_3 = b_3 = b_4 = c_3 = c_4 = 0$. Thus Equation (3.a) simplifies to become

$$0 \;=\; a_2 \mathcal{J}_x \mathcal{J}_y w + a_5 \mathcal{J}_{xy} w + b_2 \mathcal{J}_y \mathcal{J}_w x + b_5 \mathcal{J}_{yw} x + c_2 \mathcal{J}_w \mathcal{J}_x y + c_5 \mathcal{J}_{wx} y \,.$$

Applying $\mathcal{J}_{xy}$ then yields

$$\begin{aligned} 0 &= a_5 \mathcal{J}_{xy}^2 w + b_5 \mathcal{J}_{xy} \mathcal{J}_{yw} x + c_5 \mathcal{J}_{xy} \mathcal{J}_{wx} y \\ &= (a_5 \mathcal{J}_{xy}^2 + \tfrac{1}{4}(b_5 + c_5) \mathcal{J}_x \mathcal{J}_y) w \\ &= (a_5 - \tfrac{1}{2}(b_5 + c_5)) \mathcal{J}_{xy}^2 w \,. \end{aligned}$$

This shows $a_5 = \frac{1}{2}(b_5 + c_5)$ or $a_5 = b_5 = c_5$. By Equation (3.b), we have $a_5 = b_5 = 0$. Taking the inner product with x, y, and w then yields, respectively $b_2 = 0$, $c_2 = 0$, and $a_2 = 0$, which completes the proof of Assertion (2).

To prove Assertion (3), we compute:

$$\begin{aligned} & e_5 + f_5 + g_5 = \mathcal{J}_{xy} w + \mathcal{J}_{yw} x + \mathcal{J}_{wx} y \\ = \;& \tfrac{1}{2}\{\mathcal{R}(w,x)y + \mathcal{R}(w,y)x + \mathcal{R}(x,y)w + \mathcal{R}(x,w)y + \mathcal{R}(y,w)x + \mathcal{R}(y,x)w\} \\ = \;& 0 \,. \end{aligned}$$

Assertion (4) is immediate from Assertion (2). $\square$

The following example in signature $(8, 6)$ was motivated by the proof of Lemma 3.1. It shows the inequality $\dim(V) \leq 13$ in Theorem 1.6 is sharp. The proof is a computer assisted calculation which we omit in the interest of brevity. Details are available upon request from the first author.

Lemma 3.2. *Let $\{e_1, \ldots, e_4, \bar{e}_1, \ldots, \bar{e}_4, \tilde{e}_1, \ldots, \tilde{e}_4, f_1, f_2\}$ be a basis for a 14 dimensional vector space V. Relative to this basis, define an inner product $\langle \cdot, \cdot \rangle$ and an algebraic curvature tensor A on V whose non-zero components are given up to the usual $\mathbb{Z}_2$ symmetries by:*

$$\begin{aligned} & \langle e_1, e_2 \rangle = \langle e_3, e_4 \rangle = \langle \bar{e}_1, \bar{e}_2 \rangle = \langle \bar{e}_3, \bar{e}_4 \rangle = \langle \tilde{e}_1, \tilde{e}_2 \rangle = \langle \tilde{e}_3, \tilde{e}_4 \rangle = 1, \\ & \langle f_1, f_1 \rangle = \langle f_2, f_2 \rangle = -\tfrac{1}{2}, \qquad \langle f_1, f_2 \rangle = \tfrac{1}{4}, \\ & A(e_1, \tilde{e}_1, \tilde{e}_1, e_3) = A(e_1, \bar{e}_1, \bar{e}_1, e_4) = 1, \;\; A(\bar{e}_1, e_1, e_1, \bar{e}_3) = A(\bar{e}_1, \tilde{e}_1, \tilde{e}_1, \bar{e}_4) = 1, \\ & A(\tilde{e}_1, e_1, e_1, \tilde{e}_3) = A(\tilde{e}_1, \bar{e}_1, \bar{e}_1, \tilde{e}_4) = 1, \\ & A(e_1, \bar{e}_1, \tilde{e}_1, f_1) = A(e_1, \tilde{e}_1, \bar{e}_1, f_1) = A(\bar{e}_1, \tilde{e}_1, e_1, f_2) = A(\bar{e}_1, e_1, \tilde{e}_1, f_2) = -\tfrac{1}{2}. \end{aligned}$$

Then $\mathfrak{M} := (V, \langle \cdot, \cdot \rangle, A)$ is Jacobi–Tsankov, $\mathfrak{M}$ has signature $(8, 6)$, and $\mathfrak{M}$ is not 2-step Jacobi nilpotent.

4. The classification of indecomposable 2-step Jacobi nilpotent algebraic curvature tensors

In this section, we prove Theorem 1.9. The following Lemma shows that Assertion (3) implies Assertion (2) in Theorem 1.9.

Lemma 4.1. *Let $\mathfrak{M}$ be as in Definition 1.8. Then $\mathfrak{M}$ is indecomposable and 2-step skew-curvature nilpotent.*

Proof. Suppose there is a non-trivial decomposition $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2$. This would then induce a non-trivial decomposition of (W, A_W). Since (W, A_W) is assumed indecomposable, either $W \subset V_1$ or $W \subset V_2$; we suppose without loss of generality that $W \subset V_1$. Since $V_2 \perp V_1$ and $W \subset V_1$, $V_2 \perp W$ so $V_2 \subset W^\perp = W$. Thus V_2

is totally isotropic which is false. This shows $\mathfrak{M}$ is indecomposible. The following argument shows that $\mathfrak{M}$ is 2-step curvature nilpotent. Choose a basis $\{e_i\}$ for W and choose a basis $\{\bar{e}_i\}$ for $\bar{W}$ so the only non-zero components of the inner product are $\langle e_i, \bar{e}_j\rangle = \delta_{ij}$. We have

$$\mathcal{A}(e_i, e_j)e_k = \textstyle\sum_l A_W(e_i, e_j, e_k, e_l)\bar{e}_l,$$

while $\mathcal{A}(e_i, e_j)e_k = 0$ if any entry belongs to $\bar{W}$. □

We now show Assertion (2) implies Assertion (1) in Theorem 1.9.

Lemma 4.2. *Let $\mathfrak{M} := (V, \langle\cdot,\cdot\rangle, A)$. If $\mathfrak{M}$ is 2-step skew-curvature nilpotent, then $\mathfrak{M}$ is 2-step Jacobi nilpotent.*

Proof. Suppose A is a 2-step skew-curvature nilpotent algebraic curvature tensor. Then $\mathcal{A}(x_1, x_2)\mathcal{A}(x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$. Hence

$$\begin{aligned} 0 &= -\langle \mathcal{A}(x_1, x_2)\mathcal{A}(x_3, x_4)x_4, x_2\rangle = -\langle \mathcal{A}(x_1, x_2)\mathcal{J}(x_4)x_3, x_2\rangle \\ &= -\langle \mathcal{J}(x_4)x_3, \mathcal{A}(x_1, x_2)x_2\rangle = \langle \mathcal{J}(x_4)x_3, \mathcal{J}(x_2)x_1\rangle \\ &= \langle \mathcal{J}(x_2)\mathcal{J}(x_4)x_3, x_1\rangle\,. \end{aligned}$$ □

Before completing the proof of Theorem 1.9, we must establish a technical result.

Lemma 4.3. *Let $\mathfrak{M} := (V, \langle\cdot,\cdot\rangle, A)$. Suppose that $\mathcal{J}(x)y = 0$ for all $x \in V$. Then $A(x_1, x_2, x_3, y) = 0$ for all $x_i \in V$.*

Proof. We compute:

$$\begin{aligned} & A(x_1, x_2, x_3, y) + A(x_1, x_3, x_2, y) = 2\langle \mathcal{J}(x_2, x_3)x_1, y\rangle \\ = \quad & 2\langle x_1, \mathcal{J}(x_2, x_3)y\rangle = 0\,. \end{aligned}$$

Consequently $A(x_1, x_2, x_3, y) = -A(x_1, x_3, x_2, y)$ for all $x_i \in V$. This implies

$$\begin{aligned} 0 \quad &= \quad A(x_1, x_2, x_3, y) + A(x_2, x_3, x_1, y) + A(x_3, x_1, x_2, y) \\ &= \quad A(x_1, x_2, x_3, y) - A(x_2, x_1, x_3, y) - A(x_1, x_3, x_2, y) \\ &= \quad A(x_1, x_2, x_3, y) + A(x_1, x_2, x_3, y) + A(x_1, x_2, x_3, y) \\ &= \quad 3A(x_1, x_2, x_3, y)\,. \end{aligned}$$ □

We complete our discussion by showing that Assertion (1) implies Assertion (3) in Theorem 1.9. Suppose that $\mathfrak{M}$ is indecomposable and that $\mathfrak{M}$ is 2-step Jacobi nilpotent. Set

$$\bar{W} := \mathrm{Span}_{v_1, v_2 \in V}\{\mathcal{J}(v_1)v_2\} \quad \text{and} \quad U := \{v \in V : \mathcal{J}(v_1)v = 0\ \forall v_1 \in V\}\,.$$

Then by assumption, $\bar{W} \subset U$. Furthermore, by Lemma 4.3, $A(v_1, v_2, v_3, v_4) = 0$ if any of the $v_i \in U$. Choose a complementary subspace W_1 so that $V = U \oplus W_1$.

If $\bar{w} \in \bar{W}$, then $\bar{w} = \sum_j \mathcal{J}(x_j)y_j$. Thus if $u \in U$,

$$\langle \bar{w}, u\rangle = \langle \textstyle\sum_j \mathcal{J}(x_j)y_j, u\rangle = \sum_j \langle y_j, \mathcal{J}(x_j)u\rangle = 0\,. \tag{4.a}$$

Since the metric is non-degenerate, there must exist $\tilde{w} \in W_1$ so $\langle \tilde{w}, \bar{w}\rangle \neq 0$. Thus the natural map $W_1 \to \bar{W}^*$ defined by $\langle\cdot,\cdot\rangle$ is surjective. Let $\{\bar{w}_1, ..., \bar{w}_k\}$ be a basis for $\bar{W}$. Choose elements $\{\tilde{w}_1, ..., \tilde{w}_k\}$ in W_1 so

$$\langle \tilde{w}_i, \bar{w}_j\rangle = \delta_{ij}\,.$$

Suppose that $\{\tilde{w}_1, ..., \tilde{w}_k\}$ do not span W_1. We may then choose $0 \neq \tilde{w} \in W_1$ so that $\tilde{w} \perp \bar{W}$. Since $\tilde{w} \notin U$, there exists y so that $\mathcal{J}(y)\tilde{w} \neq 0$. Choose $z \in V$ so

$$0 \neq \langle \mathcal{J}(y)\tilde{w}, z\rangle = \langle \tilde{w}, \mathcal{J}(y)z\rangle\,.$$

This contradicts the fact that $\tilde{w} \perp \bar{W}$. Thus $\{\tilde{w}_1, ..., \tilde{w}_k\}$ is a basis for W_1. We set

$$w_i := \tilde{w}_i - \tfrac{1}{2}\textstyle\sum_j \langle \tilde{w}_i, \tilde{w}_j\rangle \bar{w}_j \quad \text{and} \quad W := \mathrm{Span}\{w_i\}\,.$$

Then the relations of Equation (1.a) are satisfied. Furthermore,

$$V = U \oplus W\,.$$

Let $\{\bar{w}_1, ..., \bar{w}_k, \tilde{u}_1, ..., \tilde{u}_l\}$ be a basis for U. By Equation (4.a), $\langle \bar{w}_i, \tilde{u}_j\rangle = 0$. Set

$$u_i := \tilde{u}_i - \textstyle\sum_j \langle w_j, \tilde{u}_i\rangle \bar{w}_j\,.$$

We then have $\langle u_i, w_i\rangle = \langle u_i, \bar{w}_i\rangle = 0$. Let $T := \operatorname{Span}\{u_i\}$. Then:

$$(V, \langle\cdot,\cdot\rangle, A) = (W \oplus \bar{W}, \langle\cdot,\cdot\rangle|_{W\oplus\bar{W}}, A|_W \oplus 0) \;\oplus\; (T, \langle\cdot,\cdot\rangle|_T, 0)\,.$$

Since $(V, \langle\cdot,\cdot\rangle, A)$ is indecomposable, $T = \{0\}$. □

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References

[1] N. Blažić, *Natural curvature operators of bounded spectrum*, In press J. Diff. Geo. and Applications.

[2] N. Blažić, N. Bokan, and P. Gilkey, *A Note on Osserman Lorentzian manifolds*, Bull. London Math. Soc., **29** (1997), 227–230.

[3] N. Blažić, N. Bokan, P. Gilkey, and Z. Rakić, *Pseudo-Riemannian Osserman manifolds*, Balkan J. Geom. Appl. **2** (1997), 1–12.

[4] N. Bokan, *On the complete decomposition of curvature tensors of Riemannian manifolds with symmetric connection*, Rendiconti del Circolo Matematico di Palermo **XXIX** (1990), 331–380.

[5] M. Brozos-Vázquez and P. Gilkey, *Manifolds with commuting Jacobi operators*, preprint math.DG/0507554.

[6] Q. S. Chi, *A curvature characterization of certain locally rank one symmetric spaces*, J. Differential Geom., **28** (1988), 187–202.

[7] J. C. Díaz-Ramos, E. García-Río, and R. Vázquez-Lorenzo, *New examples of Osserman metrics with nondiagonalizable Jacobi operators*, in press J. Diff. Geo. and Applications.

[8] E. García–Río, D. Kupeli, and M. E. Vázquez-Abal, *On a problem of Osserman in Lorentzian geometry*, Differential Geometry and its Applications **7** (1997), 85–100.

[9] P. Gilkey, **Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor**, World Scientific (2002).

[10] P. Gilkey, R. Ivanova, and T. Zhang, *Higher order Jordan Osserman, Pseudo-Riemannian manifolds*, Class and Quantum Gravity **19** (2002), 4543-4551.

[11] P. Gilkey and S. Nikčević, *Complete curvature homogeneous pseudo-Riemannian manifolds*, Classical and Quantum Gravity **21** (2004), 3755–3770.

[12] Y. Nikolayevsky, *Osserman manifolds of dimension 8*, Manuscr. Math. **115** (2004), 31–53.

[13] Y. Nikolayevsky, *Osserman Conjecture in dimension* $n \neq 8, 16$, Mat. Annalen **331** (2005), 505–522.

[14] R. Osserman, *Curvature in the eighties*, Amer. Math. Monthly **97** (1990), 731–756.

[15] J. M. Overduin and P. S. Wesson, *Kaluza-Klein gravity*, Phys. Rep. **283** (1997), 303–378.

[16] V. Sahni and Y. Shtanov, *Bouncing braneworlds*, Physics Letters B, **557** (2003), 1–6.

[17] Y. Tsankov, *A characterization of* n*-dimensional hypersurface in* $\mathbb{R}^{n+1}$ *with commuting curvature operators*, *Banach Center Publ.* **69** (2005), 205–209.

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